Classical Methods to solve differential equations

First order differential equations

- Separable
- Linear: $y' + P(x)y = Q(x)$
- Exact
- Substitution Methods
  - Homogeneous
  - Bernoulli
  - *Jazz*

Second order differential equations

- Reducible
  - No-y: $F(x, y', y'') = 0$
  - No-x: $F(y, y', y'') = 0$
- Linear: $y'' + P(x)y' + Q(x)y = F(x)$
  - Linear with constant coefficients homogeneous

$$ay'' + by' + cy = 0$$

  - Linear with constant coefficients non-homogeneous

$$ay'' + by' + cy = F(x)$$

Variation of Parameters

This method is just a formula. In 5 steps:

Step#1 Solve the homogeneous part, and collect y_1 and y_2.

Step#2 Compute the Wronskian $W(y_1, y_2)$.

Step#3 Compute the following:

$$A(x) = -\frac{1}{a}\int \frac{y_2(x)F(x)}{W(y_1, y_2)}dx + A$$
$$B(x) = \frac{1}{a}\int \frac{y_1(x)F(x)}{W(y_1, y_2)}dx + B$$

The solution is given by the formula

$$y = A(x)y_1(x) + B(x)y_2(x)$$

Example. Solve the following differential equation

$$y'' - 3y' - 4y = 4x^2 - 1$$

Step#1 Solve the homogeneous part: $y'' - 3y' - 4y = 0$.

$$r^2 - 3r - 4 = 0$$

$$\begin{aligned} r &= \frac{3 \pm \sqrt{9 - 4(-4)}}{2} \\ &= \frac{3 \pm 5}{2} = \{-1, 4\} \end{aligned}$$

We have computed $y_1 = e^{-x}$ and $y_2 = e^{4x}$.

Step#2 Compute the Wronskian

$$W(y_1, y_2) = \begin{vmatrix} e^{-x} & e^{4x} \\ -e^{-x} & 4e^{4x} \end{vmatrix} = 4e^{4x}e^{-x} + e^{4x}e^{-x} = 5e^{3x}$$

Step#3 Compute the functions $A(x)$ and $B(x)$.

$$\begin{aligned} A(x) &= -\frac{1}{a}\int \frac{y_2 F}{W}dx = -\int \frac{e^{4x}(4x^2-1)}{5e^{3x}}dx \\ B(x) &= \frac{1}{a}\int \frac{y_1 F}{W}dx = \int \frac{e^{-x}(4x^2-1)}{5e^{3x}}dx \end{aligned}$$

Let me help you with one of those:

$$\int \frac{e^{4x}(x^2-1)}{e^x}dx = \int e^x(x^2-1)dx = \underbrace{\int x^2 e^x dx}_{IbPs} - \underbrace{\int e^x dx}_{e^x}$$

$$\int x^2 e^x dx = uv - \int vdu = x^2e^x - \int 2xe^x dx$$

$$\begin{aligned} u &= x^2 & dv &= e^x dx \\ du &= 2xdx & v &= e^x \end{aligned}$$

$$= x^2e^x - \left(uv - \int vdu\right) = x^2e^x - \left(2xe^x - \int 2e^x dx\right)$$

$$\begin{aligned} u &= 2x & dv &= e^x dx \\ du &= 2dx & v &= e^x \end{aligned}$$

$$= x^2e^x - 2xe^x + 2e^x = e^x(x^2 - 2x + 2)$$

The Method of Undetermined Coefficients

This method requires NO integration. This is the way it works:

$$ay'' + by' + cy = F(x)$$

Step#1 Solve the homogeneous equation, and consider now the solution $y_c = Ay_1 + By_2$ as we learned yesterday.

Step#2 Look at the form of $F(x)$. We are able to use this method only if F is:

- A polynomial (like $F(x) = 4x^2 - 1$)
- An exponential times a polynomial (like $F(x) = e^{5x}$ or $F(x) = 5x^2e^{-6x}$ or $F(x) = (4x - 5)e^x$)
- An exponential times a polynomial times a sine or a cosine (for example $F(x) = 5xe^{-3x}sin(6x)$ or $F(x) = (3x^2 - 4x + 3)e^{45x}cos(\pi x)$)

Step#3 If F is a polynomial, then the solution of the differential equation is of the form

$$y = y_c + Y$$

where Y is a polynomial of the form $Y = x^s P_n(x)$. The polynomial P_n has the same degree as F. The value of s is "how many times 0 is a root of the characteristic equation of the homogeneous part."

Example. Use the method of undetermined coefficients to solve the differential equation

$$y'' - 3y' - 4y = 4x^2 - 1$$

Step#1 Solve the homogeneous part $y'' - 3y' - 4y = 0$.

$$y_c = Ae^{-x} + Be^{4x}$$

Step#2 Since $F(x) = 4x^2 - 1$ (a polynomial of degree 2), the particular solution is $Y(x) = x^s(A_0 + A_1x + A_2x^2)$. Let's find the value of s now. Question: How many times is "zero" a root of the characteristic equation? None: Then, $s = 0$

We have now decided that it must be $Y(x) = x^0(A_0 + A_1x + A_2x^2) = A_0 + A_1x + A_2x^2$

Step#3 The solution of this equation is of the form

$$\mathbf{y}=\mathbf{y}_c + Y = \underbrace{Ae^{-x} + Be^{4x}}_{y_c} + \underbrace{A_0 + A_1x + A_2x^2}_{Y}$$

Let's find the value of the undetermined coefficients A_0, A_1 and A_2.

$$\begin{aligned} y &= Ae^{-x} + Be^{4x} + A_0 + A_1x + A_2x^2 \\ y' &= -Ae^{-x} + 4Be^{4x} + A_1 + 2A_2x \\ y'' &= Ae^{-x} + 16Be^{4x} + 2A_2 \end{aligned}$$

Write the original equation, and substitute y, y' and y'' with the expressions above

$$\begin{aligned} y'' - 3y' - 4y &= 4x^2 - 1 \\ Ae^{-x} + 16Be^{4x} + 2A_2 & \\ -3\left(-Ae^{-x} + 4Be^{4x} + A_1 + 2A_2x\right) & \\ -4\left(Ae^{-x} + Be^{4x} + A_0 + A_1x + A_2x^2\right) &= 4x^2 - 1 \\ Ae^{-x} + 16Be^{4x} + 2A_2 & \\ +3Ae^{-x} - 12Be^{4x} - 3A_1 - 6A_2x & \\ -4Ae^{-x} - 4Be^{4x} - 4A_0 - 4A_1x - 4A_2x^2 &= 4x^2 - 1 \\ 2A_2 - 3A_1 - 6A_2x - 4A_0 - 4A_1x - 4A_2x^2 &= 4x^2 - 1 \\ -4A_2x^2 + (-6A_2 - 4A_1)x + (2A_2 - 3A_1 - 4A_0) &= 4x^2 - 1 \end{aligned}$$

Look now only to the coefficients that go with the x's

$$\begin{cases} -4A_2 & = 4 \\ -6A_2 - 4A_1 & = 0 \\ 2A_2 - 3A_1 - 4A_0 & = -1 \end{cases}$$

$$\begin{cases} A_2 & = -1 \\ A_1 & = 3/2 \\ A_0 & = -11/8 \end{cases}$$

The solution of this differential equation has the form

$$y = Ae^{-x} + Be^{4x} + \left(-\tfrac{11}{8} + \tfrac{3}{2}x - x^2\right)$$