Example. Find orthogonal trajectories of the family of curves given by the equation

$$y^2 = \frac{x^3}{k - x}$$

First step, write it in the form $F(x, y) = k$

$$\begin{aligned} k - x &= \frac{x^3}{y^2} \\ k &= \underbrace{x + \frac{x^3}{y^2}}_{F(x,y)} = x + x^3 y^{-2} \end{aligned}$$

Do implicit differentiation to find the differential equation that gives that family:

$$\begin{aligned} 0 &= 1 + 3x^2 y^{-2} - 2x^3 y^{-3} y' \\ &= 1 + 3\frac{x^2}{y^2} - 2\frac{x^3}{y^3} y' \end{aligned}$$

Switch y' with a $-1/y'$.

$$\begin{aligned} 0 &= 1 + 3\frac{x^2}{y^2} + 2\frac{x^3}{y^3} \cdot \frac{1}{y'} \\ -2\frac{x^3}{y^3} \cdot \frac{1}{y'} &= 1 + 3\frac{x^2}{y^2} = \frac{y^2 + 3x^2}{y^2} \\ \frac{y^3}{-2x^3} y' &= \frac{y^2}{y^2 + 3x^2} \\ y' &= \frac{-2x^3 y^2}{y^3(y^2 + 3x^2)} = -2\frac{x^3}{y(y^2 + 3x^2)} \end{aligned}$$

$$\begin{aligned} y' &= -2\left(\frac{x}{y}\right) \cdot \left(\frac{x^2}{y^2 + 3x^2}\right) \\ y' &= -2\frac{x}{y}\left(\frac{1}{\frac{y^2}{x^2} + 3}\right) \end{aligned}$$

This is a homogeneous equation. Do $v = y/x$, $y' = v + xv'$. And solve.

Example. The normal and the line throught the point and the origin form an isosceles triangle.

$$|y_0|\sqrt{1 + (y_0')^2} = \sqrt{x_0^2 + y_0^2}$$

$$
\begin{aligned}
y^2(1+(y')^2) &= x^2+y^2 \\
y^2+y^2(y')^2 &= x^2+y^2 \\
y^2(y')^2 &= x^2 \\
yy' &= \pm x \\
y' &= \pm\frac{x}{y}
\end{aligned}
$$

Example. Study the model given by the equation

$$\frac{dP}{dt} = (P+2)(P-2)^2$$

Singular solutions at $P = 2$ and $P = -2$.
However, $P = -2$ makes no sense in terms of populations.
0—2—∞
Pick a point between 0 and 2 (say, 1). Evaluate: $(1+2)(1-2)^2 = 3 > 0$.
Pick a point after 2 (say, 3). Evaluate: $(3+2)(3-2)^2 = 5 > 0$.
Sketch the corresponding slope field.

Example. The time rate of change of a rabbit population P is proportional to the square root of P. At time $t = 0$ months, the population numbers 100 rabbits and is increasing at the rate of 20 rabbits per month. How many rabbits will there be one year later?

$$\frac{dP}{dt} = k\sqrt{P}$$

$$P(0) = 100, \quad P'(0) = +20$$

From this information and the given equation, we find that

$$
\begin{aligned}
20 &= k\sqrt{100} \\
k &= 2
\end{aligned}
$$

So we have the differential equation: $\frac{dP}{dt} = 2\sqrt{P}$. Let's solve it:

$$
\begin{aligned}
\int \frac{dP}{\sqrt{P}} &= 2\int dt \\
2\sqrt{P} &= 2t + C \\
\sqrt{P} &= t + C
\end{aligned}
$$

Let's find out the value of C. We use the initial condition $P(0) = 100$ for that.

$$
\begin{aligned}
\sqrt{100} &= 0 + C \\
C &= 10
\end{aligned}
$$

Our final equation is then $\sqrt{P} = t + 10$.
How many rabbits one year later? ($t = 12$). Solve for P in the equation

$$\sqrt{P} = 22$$
$$P = 22^2 = 484$$

Solution. There will be 484 rabbits after one year.