1 An optimization problem

The Acme Widget Company has found that if widgets are priced at \$170, then 16000 will be sold. They have also found that for every increase of \$14, there will be 400 fewer widgets sold. The cost of producing each widget is \$93.5. The fixed costs for the Acme Widget Company are \$38000.

1. What is the cost function?
2. What is the revenue function?
3. The profit function?
4. What is the optimal quantity to maximize profit? (They are interested in quantity q)

The technique is always the same:

Demand - *Revenue* - *Cost* - *Profit* - Interval - Find global max of Profit in interval.

Let's go sentence by sentence:

The Acme Widget Company has found that if widgets are priced at \$170, then 16000 will be sold.

$p = 170$,$q = 16000$ This gives me a point: $(170, 16000)$

They have also found that for every increase of \$14, there will be 400 fewer widgets sold.

$p = 170 + 14, q = 16000 - 400$. This gives us another point: $(184, 15600)$

These two sentences talk about **demand.**

What kind of function do you think this particular demand is?

$$y - y_0 = slope(x - x_0)$$
$$q - q_0 = slope(p - p_0)$$
$$q{-}16000 = slope(p - 170)$$

Let's find the slope from the two points: $\dfrac{16000 - 15600}{170 - 184} = \dfrac{400}{-14} \approx -28.57$

$$q - 16000 = -28.57(p - 170)$$

The cost of producing each widget is \$93.5. The fixed costs for the Acme Widget Company are \$38000.

$$Cost = FC + TVC(q)$$
$$C(q) = 38000 + 93.5q$$

We need the revenue

$$Revenue = \overbrace{Units}^{q} \times \overbrace{Price}^{p}$$
$$R(q) = q \times D(q)$$

We need the demand as a function of q.

$$\begin{aligned}
q - 16000 &= -28.57(p - 170) \\
\frac{q - 16000}{-28.57} &= p - 170 \\
p - 170 &= \frac{q - 16000}{-28.57} \\
p &= \underbrace{170 - \frac{q - 16000}{28.57}}_{D(q)}
\end{aligned}$$

Long story short:

$$\begin{aligned}
R(q) &= q\left(170 - \frac{q - 16000}{28.57}\right) \\
&= q\,(170 - 0.035q + 560.03) \\
&= q(730.03 - 0.035q) \\
R(q) &= 730.03q - 0.035q^2
\end{aligned}$$

We need the profit now:

$$\begin{aligned}
Profit &= Revenue - Cost \\
P(q) &= (730.03q - 0.035q^2) - (38000 + 93.5q) \\
&= 730.03q - 0.035q^2 - 38000 - 93.5q \\
P(q) &= 636.53q - 0.035q^2 - 38000
\end{aligned}$$

We need an interval now:

What is the smallest possible q? $(q = 0)$What is the largest possible q? $(q = 16000)$

We have now the interval $[0, 16000]$

We need at this point to find the maximum value of the function $P(q) = 636.53q - 0.035q^2 - 38000$ over the interval $[0, 16000]$.

Step#1: Candidates.

$$q = 0, q = 16000, q = 9093$$

Critical points of $P(q)$ inside of the interval.

$$\begin{aligned}
P'(q) &= 0 \\
636.53 - 0.035 \cdot 2q &= 0 \\
636.53 - 0.07q &= 0 \\
q &= \frac{636.53}{0.07} \approx 9,093.28
\end{aligned}$$

Step#2: Evaluate P on those candidates.

$$P(0) = -\$38000$$
$$P(16000) = \$1186480$$
$$P(9093) = \$2856074$$

The maximum profit will happen at the production level of 9093 widgets. At that level we make $2,856,074.something.

2 Another optimization problem.

The cost, in dollars, to produce q designer dog leashes is $C(q) = 9q + 2$, and the price-demand function, in dollars per leash, is $p = D(q) = 81 - 2q$. Find the number of leashes that will give us maximum profit.

Demand-Cost-Revenue-Profit-interval-maximize

Demand: $D(q) = 81 - 2q$.

Cost: $C(q) = 9q + 2$.

Revenue: $R(q) = q \times p = q \times D(q) = q(81 - 2q) = 81q - 2q^2$.

Profit: $P(q) = R(q) - C(q) = (81q - 2q^2) - (9q + 2) = 81q - 2q^2 - 9q - 2 = 72q - 2q^2 - 2$.

Interval: minimum number of leashes? $q = 0$

Interval: maximum number of leashes? The demand function tells you that.

$$D(q) = 0$$
$$81 - 2q = 0$$
$$q = \frac{81}{2} \approx 40.5$$

The interval is $[0, 40]$

Time to do optimization: Find the absolute max of $P(q) = 72q - 2q^2 - 2$ over the interval $[0, 40]$.

Step #1: Candidates! $q = 0, q = 40, q = 18$.

$$P'(q) = 0$$
$$72 - 4q = 0$$
$$q = \frac{72}{4} = 18$$

Step #2: Evaluate!

$$P(0) = -\$2$$
$$P(40) = -\$322$$
$$P(18) = \$646$$

Answer: We must make 18 leashes to maximize our profit. At that level of production, the profit is \$646.

The price per unit: This comes from the demand function. (price per unit, given number of units demanded)

$$D(q) = 81 - 2q$$
$$D(18) = 81 - 2 \cdot 18 = \$45$$

3 Intervals of increase/decrease and concavity.

Find the intervals of increase/decrease and concavity of the function $g(x) = 4x^3 - 6x^2 - 240x$.

Step #1: sign chart for the derivative for increase/decrease.

Step #2: sign chart for the second derivative for concavity.

$$g'(x) = 12x^2 - 12x - 240$$
$$g'(x) = 0$$
$$12x^2 - 12x - 240 = 0$$
$$12(x^2 - x - 20) = 0$$
$$12(x - 5)(x + 4) = 0$$

We've found that g is increasing at $(-\infty, -4) \cup (5, \infty)$. The function g is decreasing in $(-4, 5)$.

There is a local max at $x = -4$. There is a local min at $x = 5$.

$$g''(x) = 24x - 12$$
$$g''(x) = 0$$
$$24x - 12 = 0$$
$$x = 0.5$$

Concavity up: $g''(x) > 0$.

$$24x - 12 > 0$$
$$x > 0.5$$

Concavity down: $g''(x) < 0$.

$$24x - 12 < 0$$
$$x < 0.5$$

4 Absolute max and min

Find the absolute max and absolute min of $f(x) = 4 – 2x^2$ over the interval $-5 \leq x \leq 1$.

Step #1: Candidates! $x = -5, x = 1, x = 0$.

$$\begin{aligned} f'(x) &= 0 \\ -4x &= 0 \\ x &= 0 \end{aligned}$$

Step #2: Evaluate!

$$\begin{aligned} f(-5) &= -46 \\ f(1) &= 2 \\ f(0) &= 4 \end{aligned}$$

The absolute maximum is at $x = 0$. Its value is 4.
The absolute minimum is at $x = -5$. Its value is -46.

5 Average cost, Average Revenue, Average Profit

The cost of producing q watches is given by $C(q) = 12000 + 340q$. What is the average cost of producing 300 watches?

Average Cost is $AC(q) = \frac{C(q)}{q}$.

$$AC(300) = \frac{C(300)}{300} = \frac{12000 + 340 \cdot 300}{300} = \$380$$

6 Interpretation

If $C(x)$ is the total cost, in millions, of producing x thousand items, interpret $C'(4) = 2$.

At the production level of 4 thousand items, the total cost of production increases at the rate of 2 million dollars per each extra thousand items.

Another one:

If $R(x)$ is the revenue of selling x thousand items, intepret $R(3) = 4$ and $R'(3) = -1$.

$R(3) = 4$. The revenue obtained by selling 3 thousand items is 4 million dollars.

$R'(3) = -1$.

At the production level of 3 thousand items, the revenue of sales decreases at the rate of 1 million dollars per each extra thousand items.

7 Marginal Cost/Revenue/Profit.

It costs $C(x) = \sqrt{x}$ dollars to produce x golf balls. What is the marginal production cost to make a golf ball? What is the marginal production cost when $x = 25$? when $x = 100$? (Include units.)

$$\begin{aligned} MC(1) &= C'(1) = \tfrac{1}{2}1^{-1/2} = \tfrac{1}{2} = \$0.5/ball = 50cents/ball \\ MC(25) &= C'(25) = \tfrac{1}{2}(25)^{-1/2} = \tfrac{1}{2}\cdot\tfrac{1}{25^{1/2}} = \tfrac{1}{2}\cdot\tfrac{1}{\sqrt{25}} = \tfrac{1}{10} = \$0.1/ball = 10cents/ball \\ MC(100) &= C'(100) = \tfrac{1}{2}(100)^{-1/2} = \tfrac{1}{20} = \$0.05/ball = 5cents/ball \end{aligned}$$

(because $C'(x) = \frac{1}{2}x^{-1/2}$)

8 Punch line

A baseball team plays in he stadium that holds 52000 spectators. With the ticket price at \$11 the average attendance has been 20000. When the price dropped to \$8, the average attendance rose to 26000. Find the demand function $D(q)$, where q is the number of the spectators. Assume $D(q)$ is linear.

$p = price$, $q = spectators$

When $p = 11$, $q = 20000$. $(\overbrace{20000}^{q}, \overbrace{11}^{p})$ or $(\overbrace{11}^{p}, \overbrace{20000}^{q})$
When $p = 8$, $q = 26000$. $(26000, 8)$ or $(8, 26000)$

$$\begin{aligned} y - y_0 &= slope(x - x_0) \\ p - p_0 &= slope(q - q_0) \\ p - 11 &= slope(q - 20000) \end{aligned}$$

We need the slope $\dfrac{11-8}{20000-26000} = \dfrac{3}{-6000} = -0.0005$
Long story short:

$$p - 11 = -0.0005(q - 20000)$$

They want you to write p as a function of q. (So, solve for p in the previous equation)

$$\begin{aligned} p &= 11 - 0.0005(q - 20000) \\ p = D(q) &= 11 - 0.0005q + 10 = 21 - 0.0005q \end{aligned}$$

We need to maximize revenue.

$$\begin{aligned} Revenue &= units \times price \\ R(q) &= q \times D(q) = q(21 - 0.0005q) = 21q - 0.0005q^2 \end{aligned}$$

I am missing the... interval!!!!!

What is the minimun number of spectators? What is the maximum number of spectators?

The maximum number gotta be 52,000. We could go here with a minimum of 0 or a minimum of 20,000 spectators.

My interval here is $[20000, 52000]$

Find the maximum of $R(q) = 21q - 0.0005q^2$ on the interval $[20000, 52000]$

Step #1. Candidates! $q = 20000$, $q = 52000$, $q = 21000$

$$R'(q) = 0$$
$$21 - 0.001q = 0$$
$$q = \frac{21}{0.001} = 21000$$

Step #2. Evaluate!

$$R(20000) = \$220,000$$
$$R(21000) = \$220,500$$
$$R(52000) = -\$260,000$$

The maximum revenue occurs when the number of spectators is 21,000. The question is "what is the price of the ticket at that stage?" The demand gives you that information:

$$p = D(q) = 21 - 0.0005q$$
$$p = D(21000) = 21 - 0.0005 \cdot 21000 = \$10.50$$