Example. Find the Laplace transform of $f(x) = 3x \sin 5x$

$\mathcal{L}\{3x\sin 5x\} = 3\mathcal{L}\{x \underbrace{\sin 5x}_{g(x)}\} = 3\mathcal{L}\{xg(x)\}$

If we know the Laplace transform of $g(x)$, say $G(s)$, then then Laplace transform of $xg(x)$ is just $-G'(s)$.

$$\begin{aligned} g(x) &= \sin 5x \\ G(s) &= \frac{5}{s^2+25} \qquad (s>0) \\ G'(s) &= 5(-1)(s^2+25)^{-2}(2s) = \frac{-10s}{(s^2+25)^2} \end{aligned}$$

Summarizing:

$$\mathcal{L}\{3x\sin 5x\} = 3\mathcal{L}\{xg(x)\} = \frac{30s}{(s^2+25)^2} \qquad (s>0)$$

Example. Use the definition of Laplace transform to show that

$$\mathcal{L}\{4\sin 2x\}(s) = \frac{8}{s^2+4} \qquad (s>0)$$

$$\mathcal{L}\{4\sin 2x\}(s) = \int_0^\infty 4\sin(2x)e^{-sx}\,dx = 4\lim_{A\to\infty}\int_0^A \sin(2x)e^{-sx}\,dx$$

Let's work on the integral all by itself:

$$\int \underbrace{\sin(2x)}_{u}\underbrace{e^{-sx}}_{dv}\,dx = uv - \int v\,du$$

$$\begin{aligned} &= -\sin(2x)\frac{e^{-sx}}{s} + \frac{2}{s}\int \overbrace{\cos(2x)}^{u}\overbrace{e^{-sx}\,dx}^{dv} \\ &= -\sin(2x)\frac{e^{-sx}}{s} + \frac{2}{s}\left(uv - \int v\,du\right) \\ &= -\sin(2x)\frac{e^{-sx}}{s} + \frac{2}{s}\left(\cos(2x)\frac{e^{-sx}}{-s} - \frac{2}{s}\int \sin(2x)e^{-sx}\,dx\right) \\ &= -\sin(2x)\frac{e^{-sx}}{s} - \frac{2}{s^2}\cos(2x)e^{-sx} - \frac{4}{s^2}\int \sin(2x)e^{-sx}\,dx \end{aligned}$$

Send the last integral to the left-hand side of the equation:

$$\frac{4+s^2}{s^2}\int \sin(2x)e^{-sx}\,dx = -\frac{\sin(2x)e^{-sx}}{s} - \frac{2\cos(2x)e^{-sx}}{s^2}$$

$$\int \sin(2x)e^{-sx}\,dx = \frac{s^2}{4+s^2}\left(-\frac{\sin(2x)e^{-sx}}{s} - \frac{2\cos(2x)e^{-sx}}{s^2}\right)$$
$$= -\frac{s}{4+s^2}\sin(2x)e^{-sx} - \frac{2}{4+s^2}\cos(2x)e^{-sx}$$

The next step is to apply the FTC for this integral in the interval $[0, A]$:

$$\int_0^A \sin(2x)e^{-sx}\,dx = -\frac{s}{s^2+4}\sin(2A)e^{-sA} - \frac{2}{s^2+4}\cos(2A)e^{-sA} + \frac{2}{s^2+4}$$

We take limits when $A \to \infty$ now:

$$\lim_{A\to\infty}\int_0^A \sin(2x)e^{-sx}\,dx = \begin{cases} \text{If } s<0 & \text{the integral diverges} \\ \text{If } s>0 & \text{the integral converges to } \frac{2}{s^2+4} \end{cases}$$

We have obtained that $\mathcal{L}\{\sin(2x)\}(s) = \frac{2}{s^2+4}$, $(s > 0)$.

Example. Find the inverse Laplace transform of $\frac{1}{s^4-16}$.

$$\frac{1}{s^4-16} = \frac{1}{(s^2-4)(s^2+4)} = \frac{1}{(s-2)(s+2)(s^2+4)}$$
$$= \frac{A}{s-2} + \frac{B}{s+2} + \frac{Cs+D}{s^2+4}$$
$$= \underbrace{\frac{A}{s-2}}_{Ae^{2x}} + \underbrace{\frac{B}{s+2}}_{Be^{-2x}} + C\underbrace{\frac{s}{s^2+4}}_{\cos(2x)} + \frac{D}{2}\cdot\underbrace{\frac{2}{s^2+4}}_{\sin(2x)}$$
$$= Ae^{2x} + Be^{-2x} + C\cos(2x) + \frac{D}{2}\sin(2x)$$

Example. Find the inverse Laplace transform of

$$\frac{s^2-2s}{s^4+5s^2+4}$$

The trick here is to be able to factor the denominator:

$$s^4+5s^2+4 = 0$$
$$(s^2)^2 + 5(s^2) + 4 = 0$$
$$s^2 = \frac{-5 \pm \sqrt{25-4\cdot 4}}{2} = \frac{-5\pm 3}{2} = \{-4, -1\}$$
$$s^4+5s^2+4 = (s^2+4)(s^2+1)$$

Once factored like that, we have the following expression:

$$\begin{aligned}\frac{s^2-2s}{(s^2+4)(s^2+1)} &= \frac{As+B}{s^2+4}+\frac{Cs+D}{s^2+1}\\ &= A\frac{s}{s^2+4}+\frac{B}{2}\cdot\frac{2}{s^2+4}+C\frac{s}{s^2+1}+D\frac{1}{s^2+1}\end{aligned}$$

We have then a simple way to take the inverse transform:

$$A\cos(2x)+\frac{B}{2}\sin(2x)+C\cos x+D\sin x$$

Example. Use techniques based on the Laplace transform to solve the IVP

$$y''+3y'+2y=x, \quad y(0)=0, \quad y'(0)=2$$

$$\begin{aligned} y''+3y'+2y &= x\\ \mathcal{L}\{y''+3y'+2y\} &= \mathcal{L}\{x\}\\ \mathcal{L}\{y''\}+3\mathcal{L}\{y'\}+2\mathcal{L}\{y\} &= \frac{1}{s^2} \quad (s>0)\\ (s^2\mathcal{L}\{y\}-s\underbrace{y(0)}_{0}-\underbrace{y'(0)}_{2})+3\,(s\mathcal{L}\{y\}-y(0))+2\mathcal{L}\{y\} &= \frac{1}{s^2} \quad (s>0)\\ s^2\mathcal{L}\{y\}-2+3s\mathcal{L}\{y\}+2\mathcal{L}\{y\} &= \frac{1}{s^2} \quad (s>0)\\ (s^2+3s+2)\mathcal{L}\{y\} &= \frac{1}{s^2}+2=\frac{1+2s^2}{s^2} \quad (s>0)\\ \mathcal{L}\{y\} &= \frac{1+2s^2}{s^2(s^2+3s+2)} \quad (s>0)\end{aligned}$$

This helped us compute the Laplace transform of the solution. Time to invert it.

$$\frac{1+2s^2}{s^2(s^2+3s+2)}=\frac{1+2s^2}{s^2(s+1)(s+2)}=\frac{A}{s+1}+\frac{B}{s+2}+\frac{C}{s^2}+\frac{D}{s}$$

The solution of this equation is in the form $Ae^{-x}+Be^{-2x}+Cx+D$.

Let's compute the values of the constants:

$$\begin{aligned}\frac{1+2s^2}{s^2(s^2+3s+2)} &= \frac{A}{s+1}+\frac{B}{s+2}+\frac{C}{s^2}+\frac{D}{s}\\ 1+2s^2 &= As^2(s+2)+Bs^2(s+1)+C(s+1)(s+2)+Ds(s+1)(s+2)\\ 1+2s^2 &= As^3+2As^2+Bs^3+Bs^2+C(s^2+3s+2)+D(s^3+3s^2+2s)\\ 1+2s^2 &= (A+B+D)s^3+(2A+B+C+3D)s^2+(3C+2D)s+2C\end{aligned}$$

Pattern matching:

$$\begin{cases} 2C & = 1 \\ 3C + 2D & = 0 \\ 2A + B + C + 3D & = 2 \\ A + B + D & = 0 \end{cases}$$

$$\begin{cases} C & = \frac{1}{2} \\ D & = -\frac{3}{4} \\ 2A + B + \frac{1}{2} - \frac{9}{4} & = 2 \\ A + B - \frac{3}{4} & = 0 \end{cases}$$

Solve for A and B normally, and you get all four constants.