Test #2. Derivatives

The concept of Derivative

(also known as instantaneous rate of change)

- Compute the tangent line to the graph of a function at one point.
- Find the x–values for which the tangent line of a function is horizontal

Derivative Rules

- Power
- Exponential
- Log
- Product/Quotient
- Chain

Example. Find the tangent line to the graph of $f(x) = 4 - x^2 + 3x^7$ at $x = 1$.

$$y - \underbrace{y_0}_{f(1)} = \underbrace{slope}_{f'(1)} \cdot (x - \underbrace{x_0}_{1})$$

$x_0 = 1$ (this is given)
$y_0 = f(1) = 4 - 1 + 3 = 6$
$slope = f'(1) = 21 \cdot 1^6 - 2 \cdot 1 = 19$

$$f(x) = 4 - x^2 + 3x^7$$
$$f'(x) = 0 - 2x + 3 \cdot 7x^6 = 21x^6 - 2x$$

The solution is $y - 6 = 19(x - 1)$

Example. Find the tangent line to the graph of $f(x) = (6x - e^x)(\sqrt{x} + \ln x)$ at $x = 1$.

$$y - y_0 = slope(x - x_0)$$

$x_0 = 1$
$y_0 = f(1) = (6 \cdot 1 - e^1)(\sqrt{1} + \ln 1) = 6 - e \approx 3.28$
$slope = f'(1) = u'v + uv' = (6 - e) \cdot 1 + (6 - e)\frac{3}{2} \approx 170.65$

$$
\begin{aligned}
f(x) &= (\underbrace{6x - e^x}_{u})(\underbrace{x^{1/2} + \ln x}_{v}) \\
u &= 6x - e^x = 6 \cdot 1 - e^1 = 6 - e \\
u' &= 6 - e^x = 6 - e^1 = 6 - e \\
v &= x^{1/2} + \ln x = 1 + 0 = 1 \\
v' &= \tfrac{1}{2}x^{-1/2} + \frac{1}{x} = \tfrac{1}{2} \cdot 1 + \tfrac{1}{1} = \tfrac{3}{2} \\
f'(x) &= u'v + uv'
\end{aligned}
$$

The tangent line is $y - 3.28 = 170.65(x - 1)$.

Example. Find all the points at which the tangent line to the graph of the function $y = f(x) = 2x^3 - 3x^2 - 36x + 3$ is horizontal.

This is a two-step question:

STEP #1: Find all the x–values where the slope is zero.

$$f'(x) = 0$$

STEP #2: For each of those values, find a corresponding point (x, y).

$$
\begin{aligned}
f(x) &= 2x^3 - 3x^2 - 36x + 3 \\
f'(x) &= 2 \cdot 3x^2 - 3 \cdot 2x - 36 \\
&= 6x^2 - 6x - 36
\end{aligned}
$$

$$
\begin{aligned}
6x^2 - 6x - 36 &= 0 \\
x^2 - x - 6 &= 0
\end{aligned}
$$

$$x = \frac{1 \pm \sqrt{1^2 - 4(-6)}}{2} = \frac{1 \pm 5}{2} = \{-2, 3\}$$

For $x = -2$. We have the point $(-2, f(-2)) =$blah...

For $x = 3$. We have the point $(3, f(3)) =$blah...

Example. Find the derivative of all these spoopy functions:

- $f(x) = 99$. $f'(x) = 0$
- $f(t) = t^2$. $f'(t) = 2t$.
- $f(x) = 4 \cdot 5^x$. $f'(x) = 4 \ln 5 \cdot 5^x$
- $f(x) = \sqrt{x} - \sqrt{41}$. $f'(x) = \frac{1}{2}x^{-1/2}$
- $f(x) = (3x^5)^2$. Let's do this in two ways:
  - Chain rule: $f'(x) = 2(3x^5)^1(3 \cdot 5x^4) = 90x^5x^4 = 90x^9$

  - Using PreCalc FTW (preparation step):

$$f(x) = 3^2x^{10} = 9x^{10}$$
$$f'(x) = 9 \cdot 10x^9 = 90x^9$$

- $f(x) = \sqrt{\frac{1}{x^6}} = \sqrt{x^{-6}} = (x^{-6})^{1/2}$ (use the same techniques as before)
- $f(x) = \frac{\overbrace{x^8+2}^{u}}{\underbrace{x}_{v}}$. $f'(x) = \frac{u'v-uv'}{v^2}$.

$$u = x^8 + 2$$
$$u' = 8x^7$$
$$v = x$$
$$v' = 1$$

- $f(x) = (5 - \ln x)^{0.31}$. $f'(x) = 0.31(5 - \ln x)^{0.31-1}(0 - \frac{1}{x}) = -\frac{0.31}{x}(5 - \ln x)^{-0.69}$
- $f(x) = 13e^{9x}$. $f'(x) = 13 \cdot e^{9x} \cdot (9)$
- $f(x) = \ln(3x^2 - e^x)$. $f'(x) = \frac{6x-e^x}{3x^2-e^x}$.
- $f(x) = \ln(3x^2 - e^{5x})$. $f'(x) = \frac{6x-5e^{5x}}{3x^2-e^{5x}}$.